Taylor's Hypothesis, Hamilton's Principle, and the LANS-α Model for Computing Turbulence

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G. I. Taylor's Contributions to Lagrangian vs Eulerian Thinking about Turbulence

G. I. Taylor's Dispersion Law. An understanding of Lagrangian statistics is of great importance in the ongoing effort to develop both fundamental and practical descriptions of turbulence. For example, Prandtl's turbulent mixing length came from a Lagrangian viewpoint: It was envisioned as the turbulent analog of the mean free path of molecules in a gas. In fact, until the famous paper "Diffusion by Continuous Movements" by G. I. Taylor (1921), most turbulence theory was discussed exclusively from the Lagrangian viewpoint. However, despite the obvious importance of the Lagrangian viewpoint in turbulent combustion, reacting flows, and pollutant transport, until recently, very few measurements of Lagrangian statistics were performed at large Reynolds numbers. Instead, experimentalists performed Eulerian measurements and tried to link these measurements as best they could to the Lagrangian statistics. For example, G. I. Taylor (1921) pursued the idea originating with Prandtl and others that, "by analogy with the kinetic theory of gases," one should attempt to find ways of predicting statistical properties of the flow by taking measurements at a given point in space. One of his most influential contributions in this regard was the formula

$$\frac{d}{dt}\left\langle \left|\mathbf{X}(t)-\mathbf{X}(0)\right|^2\right\rangle = 2\int_0^t \left\langle \mathbf{u}(0)\cdot\mathbf{u}(t)\right\rangle\, dt \quad . \tag{1}$$

This formula links the Lagrangian and Eulerian statistics of turbulence. In this formula, $\langle\,\cdot\,\rangle$ denotes an appropriate statistical average and the velocity $\mathbf{u}(t)$ with assumed zero mean $\langle\mathbf{u}(t)\rangle = 0$ is defined by the fundamental formula $\dot{\mathbf{X}}(t, \mathbf{X}(0)) = \mathbf{u}(\mathbf{X}(t), t)$, as a composition of functions. This is the Eulerian velocity evaluated along the Lagrangian trajectory $\mathbf{x} = \mathbf{X}(t, \mathbf{X}(0))$ whose initial position is $\mathbf{X}(t = 0, \mathbf{X}(0)) = \mathbf{X}(0)$.

Taylor's formula is actually a definition, and it is independent of the dynamics of how a real fluid moves. For example, it does not refer to the Navier-Stokes equations. However, the formula is important because it relates two different types of experimental measurements: Its left side represents the dispersion of Lagrangian traces in the types of flows that can be measured—for example, by observing how dye spreads in a turbulent flow or how a bunch of balloons disperses in the wind.[1]

In contrast, the right side of Taylor's formula can be measured by sampling the Eulerian velocity field at a single spatial location, then averaging over time, and thereby measuring its velocity correlations.

Taylor argued that the correlation function on the right (Eulerian) side of this formula specifies the statistical properties of a stationary random function, an idea which had great influence in the subsequent development of statistical treatments in turbulence theory and elsewhere. In general, the properties of the (Lagrangian) displacement would depend on the specific trajectory under consideration. However, Taylor argued for assuming statistical homogeneity of the Eulerian velocities, which assumes that the stochastic process generating $\mathbf{u}(t)$ does not depend on the initial position $\mathbf{X}(0)$ of the trajectory. If, in addition, the stochastic process is statistically stationary, then so are the Eulerian velocity statistics. Thus, one reason for Taylor's formula to have been influential was that it made experimental measurements of Eulerian velocity at a single point seem relevant to turbulence. Eulerian measurements are much easier than Lagrangian measurements. Averaging the velocity at a fixed location, or comparing velocities at two fixed points in space at the same instant is much easier to perform than measuring the motion of fluid parcel trajectories carried in a chaotic flow then applying averaging techniques to them. However, Eulerian statistics are not equivalent to Lagrangian statistics, in general, and turbulence modeling must eventually deal with Lagrangian statistics.

G. I. Taylor's Microscale and Its Scaling Laws. G. I. Taylor (1921) introduced the length scale now called Taylor's microscale, which is intermediate between the integral scale L and the Kolmogorov dissipation scale η. The integral scale L contains the most energy on the average. Due to the nonlinearity of fluid dynamics, energy cascades from the integral scale down through the inertial range of smaller scales, until it reaches the Kolmogorov scale, $\eta = (\nu^3/\varepsilon)^{1/4}$, where viscous dissipation finally balances nonlinearity in the Navier-Stokes equations. Thus, Kolmogorov's dissipation scale signals the end of the inertial range, and it determines the average size of the smallest eddies, which are responsible for the energy dissipation rate ε effected by the viscosity ν. In contrast, Taylor's microscale λ is an intermediate length scale associated with energy dissipation rate, the viscosity and the Eulerian time-mean kinetic energy of the circulations $\overline{u^2}$ by Taylor's formula

$$\varepsilon = 15\nu \frac{\overline{u^2}}{\lambda^2} \quad . \tag{2}$$

G. I. Taylor (1921) argued that, dimensionally,

$$\left[\lambda^2\right] = \left[\overline{u^2}\Big/\overline{(\partial_x u)^2}\right] \,, \tag{3}$$

and if one assumes that viscous energy dissipation may be estimated as

$$\varepsilon \approx \overline{u^3}\Big/L = 15\nu\overline{u^2}\Big/\lambda^2 \quad , \tag{4}$$

[1] The Lagrangian statistics for the spread of such "passive tracers" was first studied quantitatively by Lewis F. Richardson (1926), in his observation of the spread of ten thousand balloons released simultaneously at the London Expo on a windy day. Each balloon contained a note asking the finder to call and tell him the location and time when the balloon came to Earth. On collecting these observations Richardson obtained the formula,

$$\frac{d}{dt}\left\langle \left|\mathbf{X}(t)\right|^2 \right\rangle \approx \left\langle \left|\mathbf{X}(t)\right|^2 \right\rangle^{2/3} \;,$$

which implies the Lagrangian dispersion increases with time as $<|\mathbf{X}(t)|^2> \approx t^3$. This famous "Richardson Dispersion Law" still challenges researchers in turbulence for many reasons, not least because it shows that the dispersion properties of turbulence are "anomalous" (non-Gaussian). This is one indication of the "intermittency" of turbulence. (In contrast, ordinary diffusion due to Gaussian random motion would yield the linear time dependence $<|\mathbf{X}(t)|^2> \approx t$ for the dispersion of particles.)

$$\lambda / L \approx Re^{-1/2} \quad , \tag{5}$$

where $Re = L^{4/3}\,\varepsilon^{1/3}/\nu$ is the Reynolds number based on the integral scale. A similar estimate yields the well-known formula

$$\eta / L \approx Re^{-3/4} \tag{6}$$

for the ratio of Kolmogorov's dissipation scale to the integral scale. Thus, at a given Reynolds number Re (at the integral scale), Taylor's microscale exceeds Kolmogorov's dissipation scale by the factor

$$\lambda / \eta \approx Re^{1/4} \quad . \tag{7}$$

A physical interpretation of Taylor's microscale has recently emerged in the context of Lagrangian-averaged computational turbulence models. In particular, the LANS-α model is parameterized by the length scale α, which is the mean correlation length of a Lagrangian trajectory with its own running time average. Remarkably, the Lagrangian-averaged dynamics of the LANS-α model achieves a balance between its modified nonlinearity and its viscous dissipation, occurring at a length scale that has precisely the same Reynolds scaling as Taylor's microscale. Before explaining this result, we need to review another of Taylor's contributions linking Lagrangian statistics to the experimental interpretation of Eulerian measurements in turbulence.

G. I. Taylor's 1938 Frozen-in Turbulence Hypothesis. G. I. Taylor (1938) made the hypothesis that, because turbulence has high power at large length scales, the advection contributed by the turbulent circulations themselves must be small, compared with the advection produced by the larger integral scales, which contain most of the energy. Therefore, in such a situation, the advection of a field of turbulence past a fixed point can be taken as being mainly due to the larger, energy containing scales. This is the frozen-in turbulence hypothesis of G. I. Taylor. Although only valid when the integral scales have sufficiently high power compared with the smaller scales, this hypothesis delivered another very convenient linkage between the Eulerian and Lagrangian viewpoints of turbulence. Taylor's hypothesis holds, provided $\overline{u^2} \ll U^2$, where $\overline{u^2}$ is a reasonable approximation for the variations of rapidly circulating quantities that are swept along in the x-direction by the larger scales in the flow and do not influence their own evolution.

G. I. Taylor made his frozen-in turbulence hypothesis in terms of the Eulerian mean flow and, since then, others have followed suit. In experiments, this substitution allows time series measured at a single point to be interpreted as spatial variations being swept along in the Eulerian mean flow. This frozen-in turbulence advects with the Eulerian mean flow; so it remembers its initial conditions for a while. For example, advection of the three components of a vector quantity $\boldsymbol{\xi}$ by a three-dimensional Eulerian mean velocity field $\overline{\mathbf{u}}$ is expressed as

$$\frac{d}{dt}\xi\big(t,\mathbf{x}(t)\big) = \frac{\partial \xi}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \xi = 0 \quad , \quad \text{along} \quad \frac{d\mathbf{x}}{dt} = \overline{\mathbf{u}} \quad . \tag{8}$$

Thus, the advected quantity $\boldsymbol{\xi}$ remembers its initial conditions, as it is being transported by the Eulerian mean velocity of the large-scale flow. This is Taylor's hypothesis. When it holds, this hypothesis allows the very useful conversion of data taken from single-point spatial measurements into their corresponding interpretation as temporal data, and vice versa. (Other approaches, such as two-point spatial measurements, must be used when the assumptions of Taylor's hypothesis break down.)

Using the Frozen-in Turbulence Hypothesis in a Turbulence Closure

Lagrangian averaging and the corresponding adaptation of Taylor's hypothesis of frozen-in turbulence circulations was used in Chen et al. (1998) to derive the closed system of Lagrangian-averaged Navier-Stokes-α (LANS-α) equations. This work treated the Lagrangian average of the exact flow as the large scale flow into which the turbulence circulations are frozen. Thus, Lagrangian averaging was first used to find a decomposition of the exact Navier-Stokes flow into its Lagrangian mean and rapidly circulating parts. Then Taylor's hypothesis was used as a closure approximation.

Lagrangian averaging of fluid equations is a standard technique, which is reviewed, for example, in Andrews and McIntyre (1978). However, Lagrangian averaging does not give closed equations. That is, it does not give equations expressed only in terms of Lagrangian-averaged evolutionary quantities. Something is always left over, which must be modeled when averaging nonlinear dynamics. This is because "the average of a product is not equal to the product of the averages," regardless of how one computes the averages. This difficulty is the Lagrangian-average version of the famous "closure problem" in turbulence.

The approach used in Chen et al. (1999) for deriving the closed Eulerian form of the inviscid convection nonlinearity in the LANS-α equations was based on combining two other earlier results. First, the Lagrangian-averaged variational principle of Gjaja and Holm (1996) was applied for deriving the inviscid averaged nonlinear fluid equations, which had been obtained by averaging Hamilton's principle for fluids over the rapid phase of their small turbulent circulations at a fixed Lagrangian coordinate. Second, the Euler-Poincaré theory for continuum mechanics of Holm, Marsden, and Ratiu (1998) was used for handling the Eulerian form of the resulting Lagrangian-averaged fluid variational principle. Next, Taylor's hypothesis of frozen-in turbulence circulations was invoked for closing the Eulerian system of Lagrangian-averaged fluid equations. Finally, the Navier-Stokes Eulerian viscous dissipation term was added, so that viscosity would cause diffusion of the newly defined Lagrangian-average momentum and proper dissipation of its total Lagrangian-averaged energy.

Gjaja and Holm had earlier derived (1996) a Lagrangian-average wave, mean-flow turbulent description, which allowed the turbulent circulations to propagate relative to the fluid. However, this Lagrangian-mean description was accomplished at the cost of adding complication in the form of self-consistent additional dynamical equations for the Lagrangian statistics of this type of turbulence. The use in Chen et al. (1998) of Taylor's hypothesis of frozen-in turbulence circulations simplified the description of the Lagrangian statistics, by assuming it is swept along by the Eulerian mean flow. Following the assumption that these Lagrangian statistics are homogeneous and isotropic, we and colleagues derived the new LANS-α turbulence equations with only one additional (constant) parameter, which is the length scale alpha.

According to the theory, alpha is the mean correlation length of a Lagrangian trajectory with its own running time average, at fixed Lagrangian label.

Practically speaking, the quantity alpha is the length scale in isotropic homogeneous turbulence at which the sweeping of the smaller scales by the larger ones first begins according to Taylor's hypothesis. That is, circulations at length scales smaller than alpha do not interact nonlinearly to create yet smaller ones in the process of their advection. However, these smaller circulations are fully present. In particular, their Lagrangian statistics contribute to the stress tensor, the inertial terms in the nonlinearity and the circulation theorem for the resulting LANS-α model.

Deriving the LANS-α Model

The motion equation for the LANS-α model is

$$\frac{\partial}{\partial t}\bar{\mathbf{v}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{v}} + \nabla \bar{\mathbf{u}}^T \cdot \bar{\mathbf{v}} - \frac{1}{2}\nabla\left(|\bar{\mathbf{u}}|^2 + \alpha^2 |\nabla \bar{\mathbf{u}}|^2\right) + \nabla \bar{p} = \nu \Delta \bar{\mathbf{v}} + \mathbf{F} \ , \tag{9}$$

with Eulerian mean velocity $\bar{\mathbf{u}}$ satisfying

$$\bar{\mathbf{v}} \equiv \bar{\mathbf{u}} - \alpha^2 \Delta \bar{\mathbf{u}} \ \text{ for a constant } \alpha^2 \text{ and } \nabla \cdot \bar{\mathbf{u}} = 0 \ . \tag{10}$$

The inviscid part of this nonlinear motion equation (its left side) emerges from the Lagrangian-averaged Hamilton's principle for ideal fluids, upon using Taylor's hypothesis of frozen-in turbulence circulations. A sketch of its derivation is given below. For full details, see Holm (1999).

Hamilton's Principle for the Euler Equations. One begins with the Lagrangian $\ell\,[\mathbf{u}, D]$ in Hamilton's principle $\delta S = 0$ with $S = \int \ell[\mathbf{u}, D]dt$ for the Euler equations of incompressible fluid motion.

$$\ell[\mathbf{u}, D] = \int \frac{1}{2} D |\mathbf{u}|^2 - p(D-1)\, d^3x \ . \tag{11}$$

This Lagrangian is the kinetic energy, constrained by the pressure p to preserve the volume element $D\ d^3x$. Conservation of the volume element $D\ d^3x$, in turn, summons the continuity equation

$$\frac{d}{dt}\left(D d^3x\right) = \left(\frac{\partial D}{\partial t} + \nabla \cdot D\mathbf{u}\right) d^3x = 0, \ \text{ along } \ \frac{d\mathbf{x}}{dt} = \mathbf{u} \ . \tag{12}$$

The constraint $D = 1$ then implies incompressibility, $\nabla \cdot \mathbf{u} = 0$, and preservation of incompressibility will determine the pressure as a Lagrange multiplier.

Varying the action yields

$$0 = \delta S = \int D\mathbf{u} \cdot \delta \mathbf{u} + \left(\frac{1}{2}|\mathbf{u}|^2 - p\right)\delta D - (D-1)\delta p\, d^3x\, dt \ . \tag{13}$$

As expected, stationarity of S under the variation of pressure δp imposes preservation of volume, $D - 1 = 0$. The variations δD and $\delta\mathbf{u}$ are given in terms of arbitrary variations of the Lagrangian trajectory $\delta\mathbf{X} = \boldsymbol{\eta}(\mathbf{x}, t)$ as

$$\delta D = -\nabla \cdot D\boldsymbol{\eta} \quad \text{and} \quad \delta\mathbf{u} = \frac{\partial\boldsymbol{\eta}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\mathbf{u} \quad . \tag{14}$$

Integration by parts and use of the continuity equation yield

$$0 = \delta S = -\int D\left[\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\mathbf{u}^T\cdot\mathbf{u} - \nabla\left(\frac{1}{2}\|\mathbf{u}\|^2 - p\right)\right]\cdot\eta + (D-1)\,\delta p d^3x dt \quad . \tag{15}$$

Cancellation between the third and fourth terms finally implies Euler's equations,

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0, \quad \text{with} \quad \nabla\cdot\mathbf{u} = 0 \,, \tag{16}$$

by vanishing of the coefficient of the arbitrary vector function $\boldsymbol{\eta}$. This is the standard derivation of Euler's equations in the Euler-Poincaré theory of Holm, Marsden, and Ratiu (1998).

Hamilton's Principle for the Lagrangian-Averaged Euler α Equations. The derivation of the Lagrangian-averaged Euler-alpha (LAE-α) equations proceeds along the same lines, except one first decomposes the fluid velocity and volume element into their Eulerian mean and fluctuating parts, as

$$D = \bar{D} + D', \quad \text{and} \quad \mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \,. \tag{17}$$

The fluctuating parts D' and $\mathbf{u}'$ of the Eulerian quantities D and $\mathbf{u}$ at a fixed point in space $\mathbf{x}$ are associated with fluctuations of the fluid parcel trajectory $\mathbf{X} = \tilde{\mathbf{X}} + \boldsymbol{\xi}(\tilde{\mathbf{X}}, t)$ around its Lagrangian mean trajectory $\tilde{\mathbf{X}}(t, \mathbf{X}_0)$. (For example, the running time average of $\mathbf{X}$ is taken at a fixed Lagrangian coordinate $\mathbf{X}_0$.) The relations between the D' and $\mathbf{u}'$ and the Lagrangian fluctuation $\boldsymbol{\xi}$, all expressed as functions of Eulerian position and time $(\mathbf{x}, t)$ are

$$D' = -\nabla\cdot\left(D\xi\right) \quad \text{and} \quad \mathbf{u}' = \frac{\partial\xi}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}} \,. \tag{18}$$

These are linearized relations, which apply for sufficiently small fluctuations. Having used these linearized relations, we need not distinguish between Eulerian and Lagrangian averaging because the difference is only relevant at higher order in the relative amplitudes of the fluctuations. The simplest variant of the Lagrangian-averaged Euler equations is derived by substituting Taylor's hypothesis in the form

$$\frac{\partial\xi}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\xi = 0, \quad \Rightarrow \quad \mathbf{u}' = -\xi\cdot\nabla\bar{\mathbf{u}} \,. \tag{19}$$

Thus, Taylor's hypothesis drastically simplifies the velocity decomposition. We now substitute this form of Taylor's hypothesis into the decomposition of fluid velocity on the Lagrangian for Euler's equations, perform the Eulerian average (in time) using the projection property $\bar{\bar{\mathbf{u}}} = \bar{\mathbf{u}}$ and then constrain the Eulerian-mean volume to be preserved ($\bar{D} - 1$). Following these steps yields the averaged

Lagrangian

$$\bar{\ell}\left[\bar{\mathbf{u}},\bar{D}\right] = \int \frac{1}{2}\bar{D}\left(\left|\bar{\mathbf{u}}\right|^2 + \left(\overline{\xi^j\xi^k}\right)\bar{\mathbf{u}}_{,j}\cdot\bar{\mathbf{u}}_{,k}\right) - \bar{p}\left(\bar{D}-1\right)d^3x \quad . \tag{20}$$

By Taylor's hypothesis, the Lagrangian statistic $(\overline{\xi^j\xi^k})$ in this expression satisfies

$$\frac{\partial}{\partial t}\left(\overline{\xi^j\xi^k}\right) + \bar{\mathbf{u}}\cdot\nabla\left(\overline{\xi^j\xi^k}\right) = 0 \ , \tag{21}$$

upon using the projection property $\bar{\bar{\mathbf{u}}} = \bar{\mathbf{u}}$ again. Consequently, homogeneous isotropic initial conditions satisfying $(\overline{\xi^j\xi^k}) = \alpha^2\delta^{jk}$ with constant α^2 are preserved by the dynamics, and the averaged Lagrangian $\bar{\ell}[\bar{\mathbf{u}}, \bar{D}]$ in Hamilton's principle $\delta\bar{S} = 0$ with $\bar{S} = \int \bar{\ell}[\bar{\mathbf{u}}, \bar{D}]dt$ for these initial conditions simplifies to

$$\bar{\ell}\left[\bar{\mathbf{u}},\bar{D}\right] = \int \frac{1}{2}\bar{D}\left(\left|\bar{\mathbf{u}}\right|^2 + \alpha^2\left|\nabla\bar{\mathbf{u}}\right|^2\right) - \bar{p}\left(\bar{D}-1\right)d^3x \quad . \tag{22}$$

Note that the constant α^2 appears in the relative kinetic energy much the same way as Taylor argued dimensionally for his microscale. That is, α^2 encodes the relative specific kinetic energies of the Eulerian mean fluid velocity $|\bar{\mathbf{u}}|^2$ and the turbulent circulations, which satisfy $\overline{|\mathbf{u}'|^2} = \alpha^2|\nabla\bar{\mathbf{u}}|^2$ because of Taylor's hypothesis of frozen-in turbulence. However, as we shall see, α is not Taylor's microscale.

Reapplying Hamilton's variational principle with this averaged Lagrangian by following the Euler-Poincaré theory, as we did before for the Euler equations, now yields the motion equation for the Lagrangian-averaged Euler-α (LAE-α) model.

$$\frac{\partial}{\partial t}\bar{\mathbf{v}} + \bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{v}} + \nabla\bar{\mathbf{u}}^T\cdot\bar{\mathbf{v}} - \frac{1}{2}\nabla\left(\left|\bar{\mathbf{u}}\right|^2 + \alpha^2\left|\nabla\bar{\mathbf{u}}\right|^2\right) + \nabla\bar{p} = 0 \ , \tag{23}$$

with

$$\bar{\mathbf{v}} \equiv \bar{\mathbf{u}} - \alpha^2\Delta\bar{\mathbf{u}} \quad \text{and} \quad \nabla\cdot\bar{\mathbf{u}} = 0 \ . \tag{24}$$

Finally, adding viscosity in Navier-Stokes form and forcing on the right side of the LAE-α model recover the LANS-α equation of motion:

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} + \nabla\mathbf{u}^T\cdot\mathbf{v} - \frac{1}{2}\nabla\left(\left|\mathbf{u}\right|^2 + \alpha^2\left|\nabla\mathbf{u}\right|^2\right) + \nabla p = \nu\Delta\mathbf{v} + \mathbf{F} \ . \tag{25}$$

Relation of LANS-α Inertial Subrange to Taylor's Microscale

The LANS-α system of equations has a variety of properties, only one of which we shall discuss here; that is, its inertial regime has two different scalings, depending on whether the circulations are either larger or smaller than alpha. In fact, its Kármán-Howarth theorem discussed in Holm (2002) implies that its kinetic energy spectrum changes from $k^{-5/3}$ for large scales, corresponding to wave numbers $k\alpha \ll 1$, to k^{-3} for small scales corresponding to wave numbers $k\alpha \gg 1$. For a dimensional argument justifying this change of scaling in the inertial regime for the LANS-α model, see Foias, Holm, and Titi (2001).

Because of this change of scaling in the LANS-α model for circulations that are larger or smaller than alpha, the inertial range is shortened for the LANS-α model. With α fixed, the wave number κ_α at the end of the second, steeper k^{-3} LANS-α inertial range is determined in Foias, Holm, and Titi (2001) to be

$$\kappa_\alpha \approx \left(\frac{1}{\alpha}\right)^{1/3} \kappa_{Ko}^{2/3} \quad . \tag{26}$$

Since the Kolmogorov dissipation wave number (κ_{Ko}) scales with integral scale Reynolds number as $\kappa_{Ko} \approx Re^{3/4}$, one finds that dissipation balances nonlinearity for the LANS-α model at $\kappa_\alpha \approx Re^{1/2}$, which is precisely the Reynolds scaling for the Taylor microscale. Thus, there is a relationship among the three progressively larger wave numbers

$$1/\alpha < \kappa_\alpha \approx Re^{1/2} < \kappa_{Ko} \approx Re^{3/4} \quad . \tag{27}$$

Shortening the inertial range for the LANS-α model to $k < \kappa_\alpha \approx Re^{1/2}$ rather than $k < \kappa_{Ko} \approx Re^{3/4}$ implies fewer active degrees of freedom in the solution for the LANS-α model, which clearly makes it much more computable than Navier-Stokes at high Reynolds numbers.

Counting Degrees of Freedom. If one expects turbulence to be "extensive" in the thermodynamic sense, then one may expect that the number of "active degrees of freedom" N_{dof} for LANS-α model turbulence should scale as

$$N_{\text{dof}}^{\alpha} \equiv \left(L\kappa_\alpha\right)^3 \approx \left(L/\alpha\right)\left(L\kappa_{Ko}\right)^2 \approx \frac{L}{\alpha} Re^{3/2} \quad , \tag{28}$$

where L is the integral scale (or domain size), κ_α is the end of the LANS-α inertial range, and $Re = L^{4/3}\varepsilon^{1/3}/v$ is the integral-scale Reynolds number (with total energy dissipation rate ε and viscosity v). The corresponding number of degrees of freedom for Navier Stokes with the same parameters is

$$N_{\text{dof}}^{\text{NS}} \equiv \left(L\kappa_{Ko}\right)^3 \approx Re^{9/4} \quad , \tag{29}$$

and one sees a possible trade-off in the relative Reynolds number scaling of the two models, provided one resolves down to the Taylor microscale. (In practice, users of the LANS-α model often find acceptable results by setting its resolution scale to be just a factor of 2 smaller than α.)

Should these estimates of the number of degrees of freedom needed for numerical simulations that use the LANS-α model relative to Navier-Stokes not be overly optimistic, the implication would be a two-thirds power scaling advantage for using the LANS-α model. That is, in needing to resolve only the Taylor microscale, the LANS-α model could compute accurate results at scales larger than α by using two decades of resolution in situations that would require three decades of resolution for the Navier-Stokes equations at sufficiently high Re.

The argument for this advantage is as follows: One factor of $(N^{\text{NS}}{}_{\text{dof}}/N^{\alpha}{}_{\text{dof}})^{1/3}$

in relative increased computational speed is gained by the LANS-α model for each spatial dimension and yet another factor (at least) for the accompanying lessened Courant-Friedrichs-Levy (CFL) time step restriction. Altogether, this would be a gain in speed of

$$\left(\frac{N_{\mathrm{dof}}^{\mathrm{NS}}}{N_{\mathrm{dof}}}\right)^{4/3} = \left(\frac{\alpha}{L}\right)^{4/3} Re \quad . \tag{30}$$

Since $\alpha/L \ll 1$ and $Re \gg 1$, the two factors in the last expressions do compete, but the Reynolds number should win out, because Re can keep increasing while the number α/L is expected to tend to a constant value, say $\alpha/L = 1/100$, at high (but experimentally attainable) Reynolds numbers, at least for simple flow geometrics. Empirical indications for this tendency were found in Chen, Foias et al. (1998, 1999a, 1999b) by comparing steady LANS-α solutions with experimental mean-velocity-profile data for turbulent flows in pipes and channels.

Thus, according to this scaling argument, a factor of 10^4 in increased speed for accurate computation of scales greater than α could occur, by using the LANS-α model at the Reynolds number for which the ratio $\kappa_{Ko}/\kappa_\alpha = 10$. An early indication of the feasibility of obtaining such factors in increased computational speed was realized in the direct numerical simulations of homogeneous turbulence reported in Chen, Holm et al. (1999), in which $\kappa_{Ko}/\kappa_\alpha \cong 4$ and the full factor of $4^4 = 256$ in computational speed was obtained using spectral methods in a periodic domain at little or no cost of accuracy in the statistics of the resolved scales. ■

Further Reading

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